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SHORTER COMMUNICATIONS

NUMERICAL EXPERIMENTS ON THE DETERMINATION OF UNSTEADY STATE TEMPERATURE DISTRIBUTION IN A SOLID PROPELLANT ROCKET MOTOR

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NOMENCLATURE

a, scale factor;
J_n, Bessel function of the first kind and order n;
t, time;
t', $\alpha \dfrac{t}{a^2}$;
$T(x, y, t)$, temperature;
$T_1(x, y)$, see equation (2c);
w, $= x+iy$, see equation (4);
x, y, spacial variables;
Y_n, Bessel function of the second kind and order.

Greek symbols

α, thermal diffusivity;
γ, separation constant;
ξ, see equation (5);
τ, see equation (2c);
η_{nm}, roots of equation (8).

INTRODUCTION

IT IS AN accepted fact that temperature plays an important role on the structural behaviour of solid propellant rocket motors.

Temperature limitations on surface-launched motors are less rigorous than those for air-launched rocket motors [1].

Parameters taken into account and extensively studied by the U.S. Navy are: environmental temperatures for surface-launched motors being transported along transcontinental shipping routes, storage temperature records obtained from coastal and inland Navy ammunition depots, etc. [1].

In flight, high Mach numbers generate severe heating conditions which are additive to those added by captive flight conditions. Thermo-elastic or thermo-viscoelastic unsteady stress analysis of the motor is certainly of basic importance from the point of view of structural integrity and operational performance of solid propellant rocket grains.

The present study deals with the determination of the unsteady temperature field in an infinitely long circular cylinder with a star shaped perforation. It is assumed that the problem is governed by Fourier's equation of heat conduction.

Such configuration is an accepted, highly simplified mathematical model of an extremely complex thermostructural problem which would require the solution of a nonlinear viscoelastic dynamic problem with coupled thermomechanical constitutive equations and exotic boundary configurations.

The approximate analytical method used in the present study enables the research engineer and applied scientist to find a unified time-dependent solution which is valid regardless the shape of the doubly connected cross section.

Admittedly other approximate methods such as the finite difference and the finite elements techniques are more general, but their accuracy is usually tested considering domains of very simple geometry, e.g. the circle or rectangle. The approach followed in this study allows for the finding of analytical solutions in other domains, thus providing an independent check of the accuracy of more general methods.

The unsteady temperature field is also evaluated using a finite element approach.*

THEORY AND DEVELOPMENT OF THE METHOD

Consider the following unsteady heat conduction problem:

$$\alpha \nabla^2 T(x, y, t) = \frac{\partial T}{\partial t} \tag{1a}$$

$$T[L_i(x, y) = 0, t] = 0 \quad (i = 1, 2) \tag{1b}$$

$$T(x, y, t)|_{t=0} = T_0 \quad \text{(a constant)}, \tag{1c}$$

where $L_i(x, y) = 0$ $(i = 1, 2)$ denotes the functional relations which define the inner and outer boundary respectively (Fig. 1). Applying the method for separation of variables one obtains the following:

$$\tau'(t) + \alpha\gamma^2\tau(t) = 0 \tag{2a}$$

$$\nabla^2 T_1(x, y) + \gamma^2 T_1(x, y) = 0 \tag{2b}$$

where γ^2 is the separation constant and

$$T(x, y, t) = T_1(x, y)\tau(t). \tag{2c}$$

The solution of equation (2a) is simply:

$$\tau(t) \sim e^{-\alpha\gamma^2 t}. \tag{3}$$

Equation (2b) may be expressed in complex variable form as:

$$4\frac{\partial^2 T_1}{\partial w \partial \bar{w}} + \gamma^2 T_1 = 0 \tag{4}$$

where $w = x+iy = R \cdot e^{i\theta}$ and $\bar{w}$ is the complex conjugate of w. Let

$$w = f(\xi); \quad \xi = r\,e^{i\theta} \tag{5}$$

*The code has been developed at Centro Atómico Bariloche, C.N.E.A.

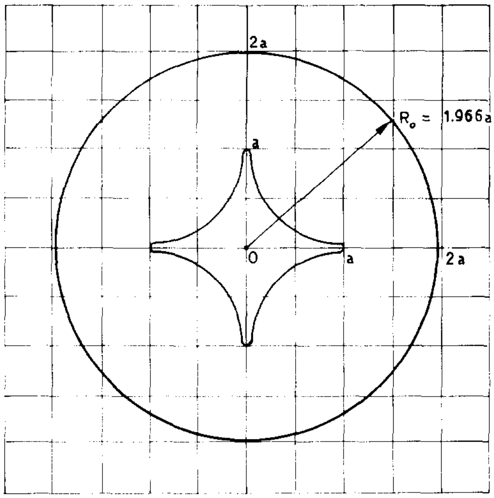

FIG. 1. Cross-section of a solid propellant rocket motor studied in the present investigation.

be the analytic function which maps the interior of an annulus in the ξ-plane onto the given domain in the w-plane.

Substituting (5) in (4) results in the transformed partial differential equation:

$$4\frac{\partial^2 T_1}{\partial \xi \partial \bar{\xi}} + \gamma^2 |f'(\xi)|^2 T_1 = 0. \quad (6)$$

Since the transformed region is now an annulus, it is convenient to express the solution of (6) as a double infinite series of cylindrical harmonics:

$$T_1(r, \theta) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} A_{nm}[J_n(\eta_{nm} r) Y_n(\eta_{nm} r_2) - J_n(\eta_{nm} r_2) Y_n(\eta_{nm} r)] \cos n\theta \quad (7)$$

where r_2 is the outer radius of the annulus, and J_n and Y_n are the Bessel functions of first and second kind respectively.

The index n denotes the order of the Bessel functions and the η_{nm}'s are the roots of the transcendental equation:

$$\begin{vmatrix} J_n(\eta_{nm} r_1) & Y_n(\eta_{nm} r_1) \\ J_n(\eta_{nm} r_2) & Y_n(\eta_{nm} r_2) \end{vmatrix} = 0 \quad (8)$$

where r_1 is the inner radius of the annulus.

Since an approximate solution is desired it will be profitable to simplify equation (7) even further. This can be done after the following considerations:

The boundary conditions (1b) and (1c) become in the ξ-plane:

$$T(r_i, \theta, t) = 0 \quad (i = 1, 2) \quad (9a)$$

$$T(r, \theta, t)|_{t=0} = T_0. \quad (9b)$$

In view of equations (9) it is reasonable to expect that isotherms in the ξ-plane will not depart drastically from concentric circumferences. Consequently the θ-dependence will be neglected.

The temperature distribution is then given by the approximate truncated expression:

$$T_1(r, \theta) \simeq \sum_{m=0}^{M} A_{0m}[J_0(\eta_{0m} r) Y_0(\eta_{0m} r_2) - J_0(\eta_{0m} r_2) Y_0(\eta_{0m} r)]. \quad (10)$$

Substituting (10) in equation (6) results in an error or residual function $\varepsilon(r, \theta)$.

Use of a suitable "weighted-residuals" approach yields the eigenvalues γ_{0m}^2. Details of the technique used have been published in the open literature [2]–[4].

Once the separation constants are known the temperature distribution in the ξ-plane is given by:

$$T(x, y, t) \simeq T(r, t) \simeq \sum_{m=0}^{M} A_{0m}[J_0(\eta_{0m} r) Y_0(\eta_{0m} r_2) - J_0(\eta_{0m} r_2) Y_0(\eta_{0m} r)] e^{-\alpha \gamma_{0m}^2 t} \quad (11)$$

where the A_{0m}'s are obtained from the initial condition (9c) and they are given by the relation [5]:

$$A_{0m} = \pi \frac{J_0(\eta_{0m} \cdot r_1)}{J_0(\eta_{0m} \cdot r_1) + J_0(\eta_{0m} \cdot r_2)}. \quad (12)$$

It is important to point out that when $M \to \infty$, *equation (11) is an exact solution if the given configuration is an annulus.*

Since (11) converges in a rapid fashion, the use of the first term is sufficient for some practical applications.

Consider now the domain shown in Fig. 1. The approximate mapping function which maps the given region onto an annulus in the ξ-plane is given by the expression [6]:

$$w = a(0.7789\xi + 0.2965\xi^{-3} - 0.0789\xi^{-7} + 0.0034\xi^{-11}). \quad (13)$$

For $r = 1$, equation (13) yields the inner contour and for $r = 2.50$ the outer boundary with an error less than 1% since for $r \gg 1$ the first term of (13) is predominant.

It must be pointed out that if the web fraction (ratio of the diameters of the circumscribing circles for the outer and inner boundaries) is closer to unity it is necessary to use a truncated Laurent expansion [7].

Finding the coefficients of the mapping function involves then solution of a system of coupled integral equations.

The next step is the calculation of the separation constants. This is done using an approach published elsewhere* [4].

For

$$\lambda = \frac{r_2}{r_1} = 2.50$$

the two lowest eigenvalues are:

$$(\gamma_{01} \cdot a) = 2.75 \quad \text{and} \quad (\gamma_{02} \cdot a) = 5.50.$$

FINITE ELEMENT SOLUTION AND COMPARISON OF RESULTS

Figure 2 shows the element distribution used in the present investigation.

Figures 3 and 4 depict a comparison of values of the dimensionless temperature parameter T/T_0 as a function of R/a and the dimensionless time scale $t' = \alpha t/a^2$. The polar variable ϕ has been taken equal to zero in all cases.

Figure 5 shows T/T_0 as a function of t' for a fixed point in space ($R/a = 1.4168$; $\phi = 0$).

The analytic solution converges slowly for small values of the temporal variable (this is the reason why the curve has been extrapolated close to the origin of the plot).

From the inspection of Figs. 3–5 one may conclude that the agreement is, in general, quite reasonable, especially if one considers that the approximate, analytic solution consists of only two terms and that the θ-dependence in the ξ-plane is disregarded in the present analysis. Admittedly the agreement is not as good for other values of ϕ, especially for the extreme case where $\phi = \pi/4$.

No claim of originality is made in the present paper, but it is hoped that present results be of some value in future investigations dealing with boundary and eigenvalue problems in domains of complicated boundary shape.

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*The separation constants are obtained solving equation (6) by means of a weighted-residuals approach [4].

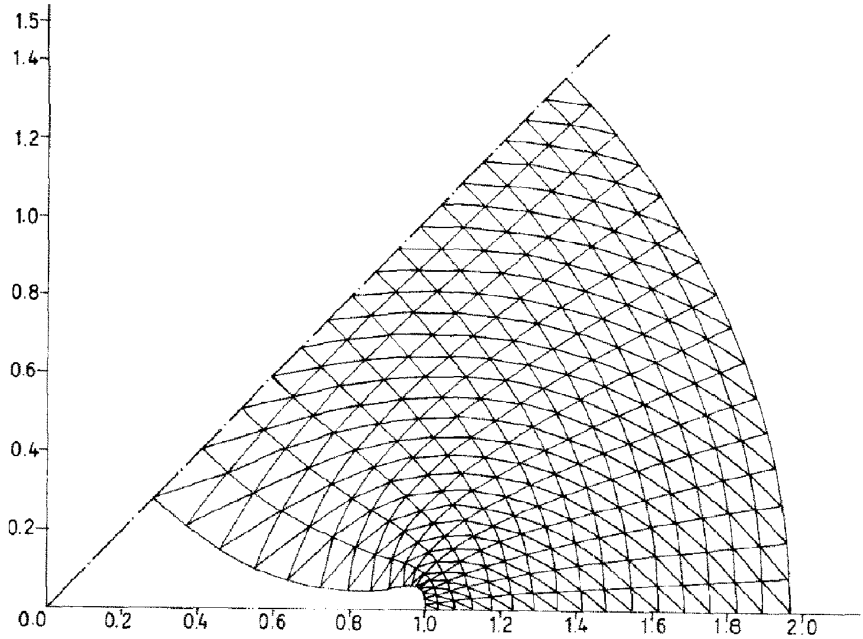

FIG. 2. Finite element distribution.

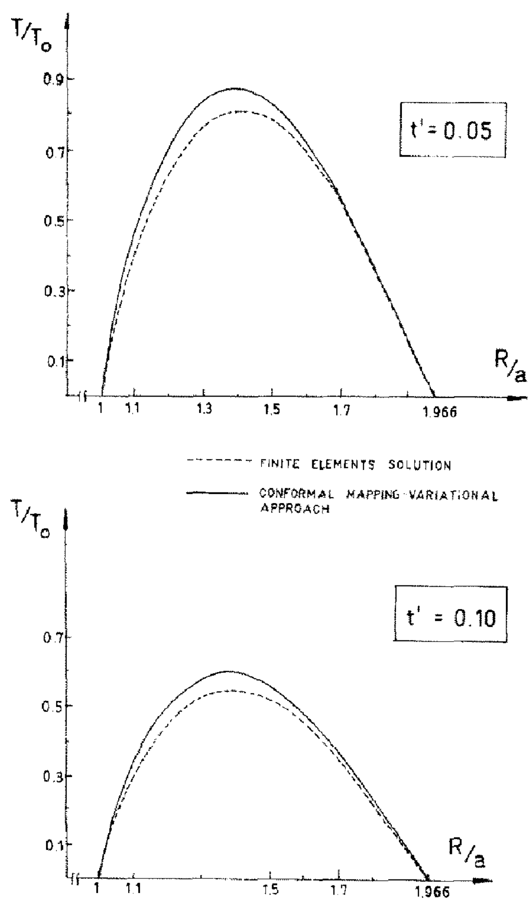

FIG. 3. Variation of T/T_0 as a function of R/a for $t' = 0.05$ and 0.10; $\phi = 0$.

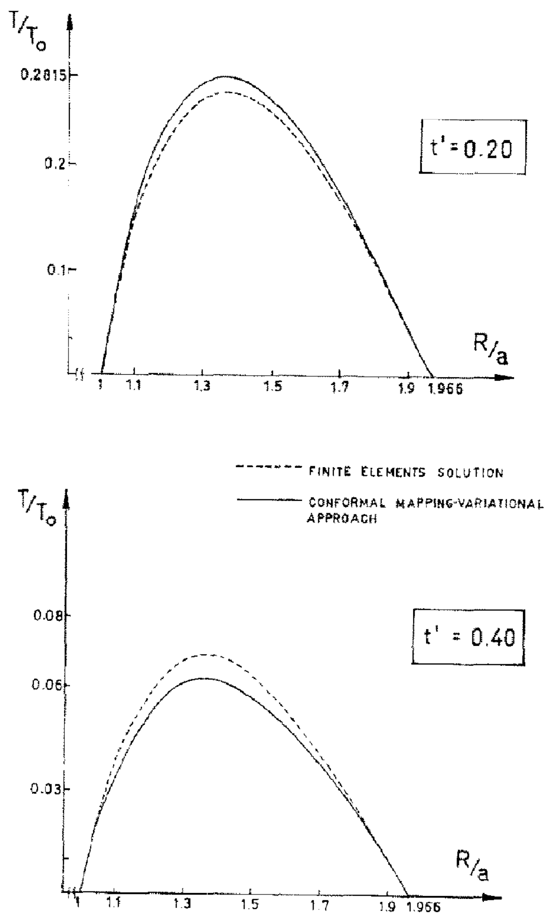

FIG. 4. Variation of T/T_0 as a function of R/a for $t = 0.20$ and 0.40; $\phi = 0$.

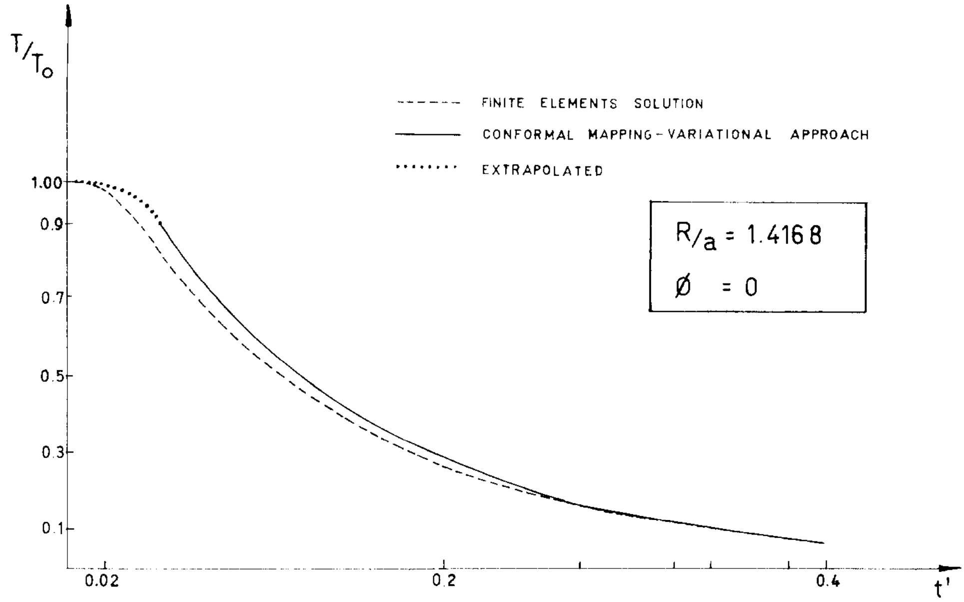

FIG. 5. Variation of T/T_0 as a function of t' for $R/a = 1.4168$; $\phi = 0$.

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